

2,3,4
But not 5

Barry A Byrne
Maths Group, Sept 2009

Al-Khwarizmi

Abu Ja'far Muhammad ibn Musa Al-Khwarizmi

Born: about 780 in Baghdad (now in Iraq)

Died: about 850



Al Jabr

*Hisab **al-jabr** w'al-muqabala*

Algorithm

First systematic treatment of
Quadratic Equations

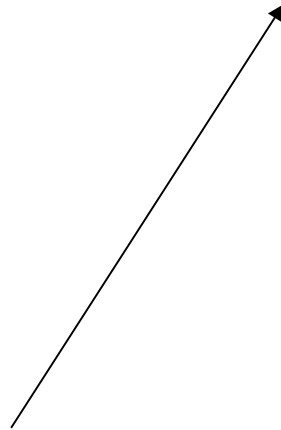
$$ax^2 + bx + c = 0$$

↓

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

↓

$$x^2 + 2\frac{b}{2a}x = -\frac{c}{a}$$



$$x^2 + 2\frac{b}{2a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

↓

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

↓

$$x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

↓

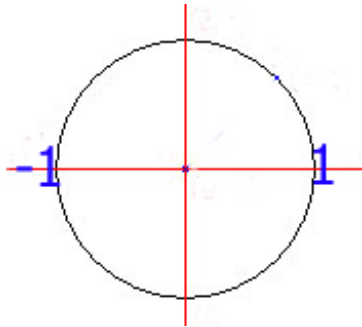
$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Things we can note...

$$x^2-2=0$$

$$x^2+1=0$$

\pm



Group with two elements multiply by $(1, -1)$ or rotate by $(0, \pi)$

We're solving the equation $x^2=1$

What is rotate by $\pi/2$?

The Fundamental Theorem of Algebra



Any polynomial with real or complex coefficients has a root in the complex plane

It follows that a polynomial of degree n has exactly n roots in the complex plane

Johann Carl Friedrich Gauss

Aged 28 in picture

Born: 30 April 1777 Died: 23 Feb 1855

The Cubic Equation

Scipione del Ferro
Born: 6 Feb 1465 in
Bologna, Italy
Died: 5 Nov 1526 in
Bologna, Italy



Nicolo Tartaglia
Born: 1500 in
Brescia,
Died: 13 Dec 1557 in
Venice

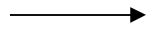


Girolamo Cardano
Born: 24 Sept 1501
in Pavia,
Died: 21 Sept 1576
in Rome

Solve $y^3 + ay^2 + by + c = 0$ $(u + v)^3 - 3uv(u + v) - (u^3 + v^3) = 0$

$$x = y - \frac{a}{3}$$

$$u + v = x$$



$$p = -3uv$$

$$x^3 + px + q = 0$$

$$q = -(u^3 + v^3)$$



$$(t - u^3)(t - v^3) = 0$$

$$t^2 + qt - \frac{p^3}{27} = 0$$

$$t = \frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

$$x = u + v = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

$$x^3+6x-20=0$$

has root 2 by inspection, but the formula gives

$$\sqrt[3]{10 + \sqrt{100 + 8}} + \sqrt[3]{10 - \sqrt{100 + 8}}$$

$$\sqrt[3]{20.392304845413263761164678049035} + \sqrt[3]{-0.392304845413263761164678049035}$$

$$2.732051 - 0.732051 = 2$$

There is a neater way

Tartaglia, there's a problem

$$x^3-15x-4=0$$

has root 4 by inspection, but the formula gives

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}}$$



Rafael Bombelli

Born: Jan 1526 in

Bologna,

Died: 1572 in Rome

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}}$$

$$(a + b\sqrt{-1})^3 = 2 + 11\sqrt{-1}$$

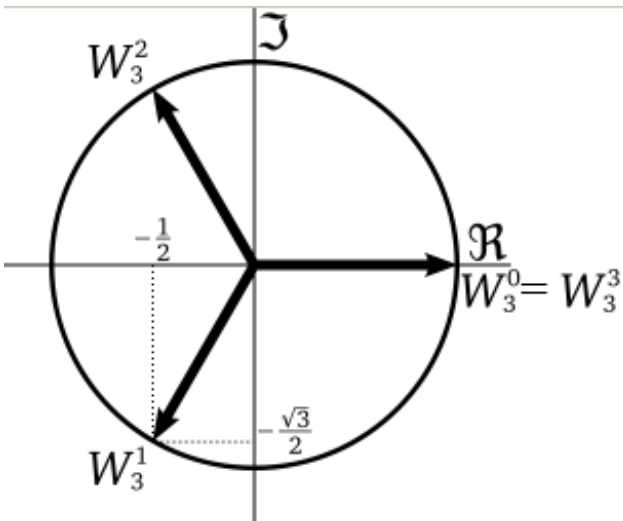
$$(a - b\sqrt{-1})^3 = 2 - 11\sqrt{-1}$$

$$a = 2, b = 1$$

$$x = 2 + \sqrt{-1} + 2 - \sqrt{-1} = 4$$

But – what about the fundamental theorem of algebra?

Three roots



Cube roots of unity

3 element group

Rotation by $2\pi/3$

$$1 \quad \xi = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \quad \xi^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

$$\xi \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \xi^2 \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

$$\xi^2 \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \xi \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Lodovico Ferrari

Born: 2 Feb 1522 in Bologna,

Died: 5 Oct 1565 in Bologna

$$x^4 + px^2 + qx + r =$$

$$(x^2 + ax + b)(x^2 - ax + c)$$

$$- a^2 + b + c = p, a(c - b) = q, bc = r$$

$$a^6 + 2pa^4 + (p^2 - 4r)a^2 - q^2 = 0$$

$$x^4 - 2x^2 + 8x - 3 = 0$$

$$a^6 - 4a^4 + 16a^2 - 64 = 0 \quad \text{cubic equation in } a^2$$

$$a^2 = 4$$

$$x = -1 \pm \sqrt{2}, -1 \pm i\sqrt{2}$$

So, is there a method to solve *by radicals*
the *quintic* equation

$$ax^5+bx^4+cx^3+dx^2+ex+f=0?$$

Sometimes “yes”

But, in general, “no”



Found solutions for special cases of the quintic

Leonhard Euler
Born: 15 April 1707 in
Basel
Died: 18 Sept 1783 in
St Petersburg



Gian Francesco Malfatti
Born: 1731 in Trento
Died: 9 Oct 1807 in
Ferrara



**Joseph-Louis
Lagrange**
**Born: 25 Jan 1736 in
Turin**
**Died: 10 April 1813
in Paris**

Analysed the methods in solving the quadratic, cubic and quartic

Polynomials on the roots

Permutations of the roots

His method, applied to the quintic,
gave a resolvent of order 6



Paolo Ruffini
Born: 22 Sept 1765 in
Valentano,
Died: 10 May 1822 in
Modena

Professor of mathematics and medicine

Built on work of Lagrange to prove
quintic not solvable by radicals

Book *General theory of equations*

Nobody understood it

or was it, bothered to read it

One gap in his proof

The algebraic solution of general equations of degree greater than four is always impossible.

*Behold a very important theorem which I believe I am able to assert (if I do not err): to present the proof of it is the main reason for publishing this volume. The immortal **Lagrange**, with his sublime reflections, has provided the basis of my proof.*

Could be credited with inventing group theory

Ahead of his time



The unhappy life of Niels Abel

Building on work of Lagrange,
proved that there was no general
method that would solve every quintic
equation

(independently of Ruffini)

Niels Henrik Abel
Born: 5 Aug 1802 in
Frindöe, Norway
Died: 6 April 1829 in
Froland



Evariste Galois
Born: 25 Oct 1811 in
Bourg La Reine
Died:
31 May 1832 in Paris

The unhappy life of Evariste Galois.

Showed that the quintic (and
higher degrees)

are in general not solvable

But explained why some are

And how to tell the difference

Invention of group theory

Permutations

a, b, c, d, e

b, a, c, d, e

e, a, b, c, d

How many permutations of n letters? $n!$

Composition of permutations is a permutation

Doing nothing is a permutation

Opposite permutation is a permutation

Therefore permutations form a group

The permutation group of n letters is called the

Symmetric Group, S_n

The Galois Group of a Polynomial

Consider the set B of polynomials in n variables that are equal to zero when the roots of the original polynomial are substituted for the variables

We should say B_K , where K is a field

There could be many, many such polynomials

The Galois group of the original polynomial is the set of permutations of its roots that doesn't change B_K

Example

Consider $ax^2+bx+c=0$, where a and b are rational numbers

Roots x_1 and x_2

We have polynomials X_1+X_2+b , X_1X_2-c

$x_1+x_2+b=0$, $x_1x_2-c=0$ for instance

Symmetric group S_2 , {do nothing, interchange roots}

Now *adjoin* $\sqrt{b^2-4c}$ to the field of rational numbers

$$x_1 - x_2 - \sqrt{b^2 - 4c} = 0$$

$$x_2 - x_1 - \sqrt{b^2 - 4c} = -2\sqrt{b^2 - 4c}$$

interchange roots is no longer in Galois group. Now $S_1 = \{1\}$

Theorem

An n degree equation is solvable in radicals
if its Galois group can be reduced in stages to S_1

That is, if S_n has the *right sort* of subgroups

Not being too technical about this, am I....?

S_3 has A_3 which has $\{1\}$

S_4 has A_4 which has V which has $\{1\}$

But S_5

S_5 , the most general Galois group of the quintic

S_5 has A_5 as its subgroup, which isn't the *right sort* of subgroup

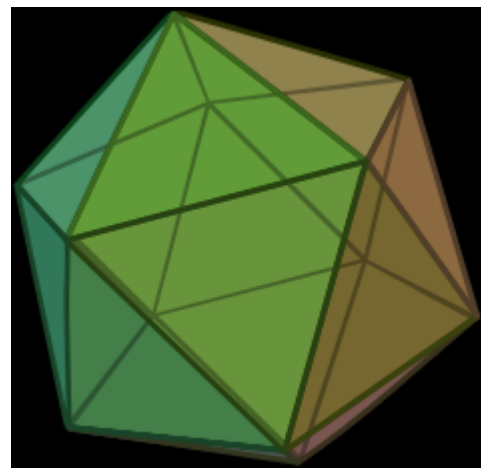
The elements of A_5 don't commute

How many permutations in S_5 ?

A_5 is the group of even permutations on five letters

A_5 contains 60 elements

Isomorphic to the **icosahedral rotation group**



So S_5 can't be reduced in Galois's way to S_1 and the general quintic is not solvable

Rule of Thumb for Quintic Equations

Possible sizes for the Galois group of an irreducible quintic equation,
120, 60, 20, 10, 5

If the Galois group has 120 or 60 elements, the quintic can't be solved

If fewer elements, it can be solved in radicals

There are only 5 permutations in the Galois group of
Vandermonde's equation

$$x^5+x^4-4x^3-3x^2+3x+1=0$$

The solutions are

“the two member periods of the cyclotomic equation of order 11”

Or to put it another way.... Vandermonde's solution

$$x = 1/5[1 + \Delta^i + \Delta^{ii} + \Delta^{iii} + \Delta^{iv}]$$

where:

$$\Delta^i = \sqrt[5]{\frac{11}{4}(89 + 25\sqrt{5} - 5\sqrt{-5 - 2\sqrt{5}} + 45\sqrt{-5 + 2\sqrt{5}})}$$

$$\Delta^{ii} = \sqrt[5]{\frac{11}{4}(89 + 25\sqrt{5} + 5\sqrt{-5 - 2\sqrt{5}} - 45\sqrt{-5 + 2\sqrt{5}})}$$

$$\Delta^{iii} = \sqrt[5]{\frac{11}{4}(89 - 25\sqrt{5} - 5\sqrt{-5 + 2\sqrt{5}} - 45\sqrt{-5 - 2\sqrt{5}})}$$

$$\Delta^{iv} = \sqrt[5]{\frac{11}{4}(89 - 25\sqrt{5} + 5\sqrt{-5 + 2\sqrt{5}} + 45\sqrt{-5 - 2\sqrt{5}})}$$

The equation $x^5 - 5x + 12 = 0$ leads to a Galois group with ten permutations. Its solutions are

$$\begin{aligned}
 x_{j+1} = & \epsilon^j \sqrt[5]{-1 + \frac{2}{5}\sqrt{5} - 3\sqrt{\frac{1}{5} - \frac{11}{125}\sqrt{5}}} \\
 & + \epsilon^{2j} \sqrt[5]{-1 - \frac{2}{5}\sqrt{5} + 3\sqrt{\frac{1}{5} + \frac{11}{125}\sqrt{5}}} \\
 & + \epsilon^{3j} \sqrt[5]{-1 - \frac{2}{5}\sqrt{5} - 3\sqrt{\frac{1}{5} + \frac{11}{125}\sqrt{5}}} \\
 & + \epsilon^{4j} \sqrt[5]{-1 + \frac{2}{5}\sqrt{5} + 3\sqrt{\frac{1}{5} - \frac{11}{125}\sqrt{5}}},
 \end{aligned}$$

where $\epsilon = \cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right)$ for $j = 0, 1, 2, 3, 4$.

ϵ is the fifth root of 1

The Galois group of the equation $x^5-2=0$

Contains 20 permutations

Its solutions are $x_j = \varepsilon^{j-1} \sqrt[5]{2}$ Where $j=1,\dots,5$

$$x^5+20x+16=0$$

Galois group has 60 permutations

$$x^5-x+1=0$$

Galois group has 120 permutations

An algebraic equation is solvable by radicals
if and only if its Galois group is *solvable*

A finite group is *solvable* (or *soluble*) if
it contains normal subgroups B_1, B_2, \dots, B_s

Such that $G \triangleright B_1 \triangleright B_2 \triangleright \dots \triangleright B_s \triangleright \{1\}$

and each B_{i-1}/B_i is abelian ($i=1, 2, \dots, s+1$)

$B_0=G$ and $B_{s+1}=\{1\}$

Note A_5 is not abelian, but its only normal subgroup is $\{1\}$

So $A_5/\{1\}$ (being $=A_5$) is not abelian

One of the great Impossibility proofs in Mathematics

The Ruffini-Abel Theorem.

And a whole branch of mathematics

Galois theory