Easily-stated Conjectures

Barry A Byrne
Wokingham U3A
Maths Group, June 2011

What is a Conjecture?

Guess

Intelligent Guess

Premier League Conjecture?

Leads to new mathematical results

"Hypothesis"

Programmes and Prizes

Hilbert's List from 1900

Clay

DARPA

23 mathematical challenges

Hilbert’s 23 problems, 1900

David Hilbert

1862-1943

1 The continuum hypothesis

2 The axioms of arithmetic are consistent

3 Given any two polyhedra of equal volume, is it always possible to cut the first into finitely many polyhedral pieces which can be reassembled to yield the second?

8 The Riemann hypothesis ("the real part of any non-trivial zero of the Riemann zeta function is ½") and other prime number problems, among them **Goldbach's conjecture** and the twin prime conjecture

Conjecture stated with difficulty

The modern statement of the Hodge conjecture is:

Hodge conjecture. Let X be a projective complex manifold.

Then every Hodge class on X is a linear combination with rational coefficients of the cohomology classes of complex subvarieties of X.

Most Famous Conjecture

aka

Fermat's Last Theorem

$$x^n + y^n = z^n$$

has no solutions in the integers for n>2

Pierre Fermat

Born: 17 Aug 1601
in Beaumont-de-Lomagne, France

Died: 12 Jan 1665 in Castres, France

Cuius rei demonstrationem mirabilem sane detexi hanc marginis exiguitas non caperet**.**

I have discovered a truly remarkable proof of this theorem which this margin is too small to contain.

n=4 Fermat <1665

n=3 Euler 1770

n<100 Sophie Germain 1820ish

for n>5 counter-examples must be at least forty digits

n=5 Dirichlet and Legendre (about 1825)

n=7 Lamé (1839)

False Proof Kummer 1843

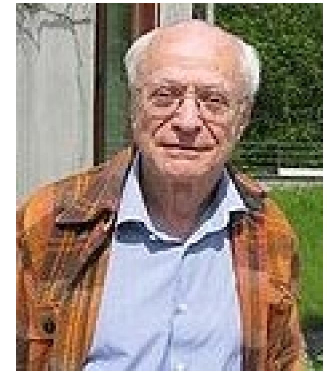
"Regular Primes" Kummer

(the only *irregular* primes <100 are 37, 59, 67)

Proof Wiles 1994

Yutaka Taniyama
1927-1958

Goro Shimura
1930-



Jean-Pierre Serre
1926-

Andrew Wiles
1953-

Shimura Taniyama Conjecture

Every rational elliptic curve is a modular form in disguise

Now known as

The Modularity Theorem

Shimura-Taniyama conjecture 1955

English 1958

André Weil rediscovered 1967

Gerhard Frey 1984

If Fermat’s conjecture false, some elliptic curves could violate Shimura-Taniyama conjecture

The “Epsilon Conjecture” Jean Pierre Serre

Ken Ribet proved the epsilon conjecture 1986

thereby proving that the TSW conjecture implied Fermat's Last

1995, Andrew Wiles, partially with the help of Richard Taylor, proved the Taniyama–Shimura–Weil conjecture for a class of elliptic curves called semistable elliptic curves, which was strong enough to prove Fermat’s Conjecture (“Last Theorem”)

Christian Goldbach

Image of
Christian Goldbach

Born: 18 March 1690 in Königsberg, Prussia (now Kaliningrad, Russia)

Died: 20 Nov 1764 in Moscow, Russia

Golbach’s Conjecture

Every even number is the sum of two (distinct) primes

False Conjecture 1

Every number is the sum of
two primes

Exercise:

Find a counter-example

11

False Conjecture 2

Every odd number is of the form
twice a square plus a prime (or 1)
$2a^2+p$

a can be zero

Test it

it works for quite a while…

Two exceptions known….

False Conjecture 2

```
>false_conjecture_2.php
3=0+3=2+1   (2)
5=0+5=2+3   (2)
7=0+7=2+5   (2)
9=2+7=8+1   (2)
11=0+11=8+3   (2)
13=0+13=2+11=8+5   (3)
15=2+13=8+7   (2)
17=0+17   (1)
19=0+19=2+17=8+11=18+1   (4)
21=2+19=8+13=18+3   (3)
23=0+23=18+5   (2)
25=2+23=8+17=18+7   (3)
27=8+19   (1)
29=0+29=18+11   (2)
31=0+31=2+29=8+23=18+13   (4)
33=2+31=32+1   (2)
35=18+17=32+3   (2)
37=0+37=8+29=18+19=32+5   (4)
39=2+37=8+31=32+7   (3)
41=0+41=18+23   (2)
43=0+43=2+41=32+11   (3)
45=2+43=8+37=32+13   (3)
49=2+47=8+41=18+31=32+17   (4)
51=8+43=32+19=50+1   (3)
53=0+53=50+3   (2)
55=2+53=8+47=18+37=32+23=50+5   (5)
57=50+7   (1)
59=0+59=18+41   (2)
61=0+61=2+59=8+53=18+43=32+29=50+11   (6)
63=2+61=32+31=50+13   (3)
65=18+47   (1)
67=0+67=8+59=50+17   (3)
69=2+67=8+61=32+37=50+19   (4)
71=0+71=18+53   (2)
73=0+73=2+71=32+41=50+23=72+1   (5)
75=2+73=8+67=32+43=72+3   (4)
77=18+59=72+5   (2)
79=0+79=8+71=18+61=32+47=50+29=72+7   (6)
81=2+79=8+73=50+31   (3)
83=0+83=72+11   (2)
85=2+83=18+67=32+53=72+13   (4)
87=8+79=50+37   (2)
89=0+89=18+71=72+17   (3)
91=2+89=8+83=18+73=32+59=50+41=72+19   (6)
```

False Conjecture 2

>php false_conjecture_2.php 5777

5777 is a counter-example!

((0, 5777), (2, 5775), (8, 5769), (18, 5759), (32, 5745), (50, 5727), (72, 5705), (98, 5679), (128, 5649), (162, 5615),
(200, 5577), (242, 5535), (288, 5489), (338, 5439), (392, 5385), (450, 5327), (512, 5265), (578, 5199), (648, 5129), (72
2, 5055), (800, 4977), (882, 4895), (968, 4809), (1058, 4719), (1152, 4625), (1250, 4527), (1352, 4425), (1458, 4319), (
1568, 4209), (1682, 4095), (1800, 3977), (1922, 3855), (2048, 3729), (2178, 3599), (2312, 3465), (2450, 3327), (2592, 31
85), (2738, 3039), (2888, 2889), (3042, 2735), (3200, 2577), (3362, 2415), (3528, 2249), (3698, 2079), (3872, 1905), (40
50, 1727), (4232, 1545), (4418, 1359), (4608, 1169), (4802, 975), (5000, 777), (5202, 575), (5408, 369), (5618, 159))

False Conjecture 2

>php false_conjecture_2.php 5993

5993 is a counter-example!

>php false_conjecture_2.php 5995

5995=8+5987=72+5923=98+5897=128+5867=338+5657=512+5483=578+5417=648+5347=722+5273=882+5113=1058+4937=1352+4643=1922+4073=2048+3947=2738+3257=3042+2953=3362+2633=3528+2467=3698+2297=4802+1193=5408+587=5832+163 (22)

Goldbach’s Conjecture

Every even number greater than 6 is the sum of two *distinct* primes

>php goldbach.php

8=3+5 (1)

10=3+7 (1)

12=5+7 (1)

14=3+11 (1)

16=3+13=5+11 (2)

18=5+13=7+11 (2)

20=3+17=7+13 (2)

22=3+19=5+17 (2)

24=5+19=7+17=11+13 (3)

26=3+23=7+19 (2)

28=5+23=11+17 (2)

30=7+23=11+19=13+17 (3)

32=3+29=13+19 (2)

34=3+31=5+29=11+23 (3)

36=5+31=7+29=13+23=17+19 (4)

38=7+31 (1)

40=3+37=11+29=17+23 (3)

42=5+37=11+31=13+29=19+23 (4)

44=3+41=7+37=13+31 (3)

46=3+43=5+41=17+29 (3)

48=5+43=7+41=11+37=17+31=19+29 (5)

50=3+47=7+43=13+37=19+31 (4)

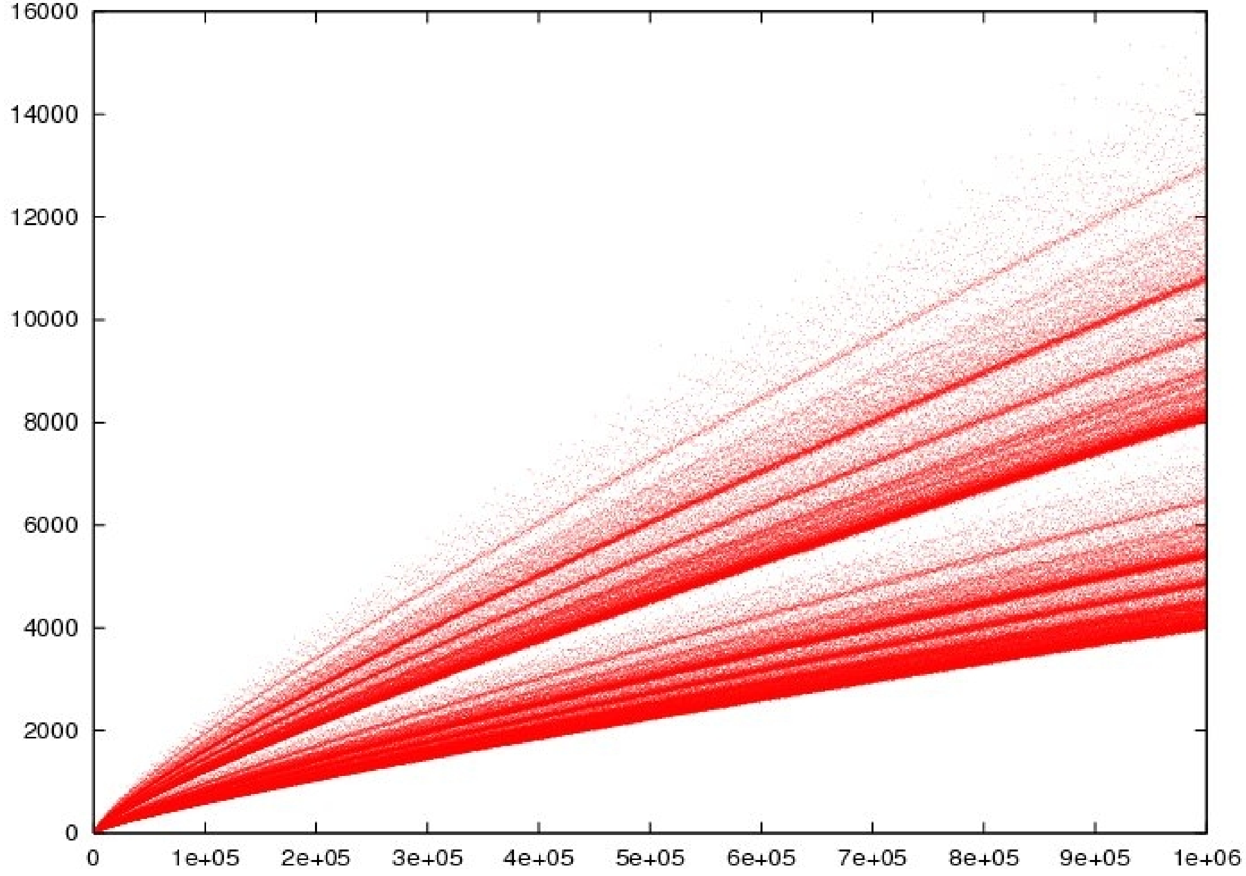
52=5+47=11+41=23+29 (3)

54=7+47=11+43=13+41=17+37=23+31 (5)

Goldbach's Conjecture

Every even number greater than 2 is the sum of two primes.

Number of pairs of primes per even number $4 \leq n \leq 1{,}000{,}000$



Goldbach's Conjecture

Chudakov, van der Corput and Estermann(1937-1938) showed that *almost all* even numbers can be written as the sum of two primes.

Chen Jingrun showed in 1973 that every *sufficiently large* even number can be written as the sum of either two primes, or a prime and a semiprime - the product of two primes—e.g.,

$100 = 23 + 7 \cdot 11$.

The Four Colour Conjecture (theorem)

Only four colours needed to colour a map

Not so “Easily Stated”

given any separation of a plane into contiguous regions, producing a figure called a *map*, no more than four colours are required to colour the regions of the map so that no two adjacent regions have the same colour.

Two regions are called adjacent only if they share a border point that is not shared with a third region.

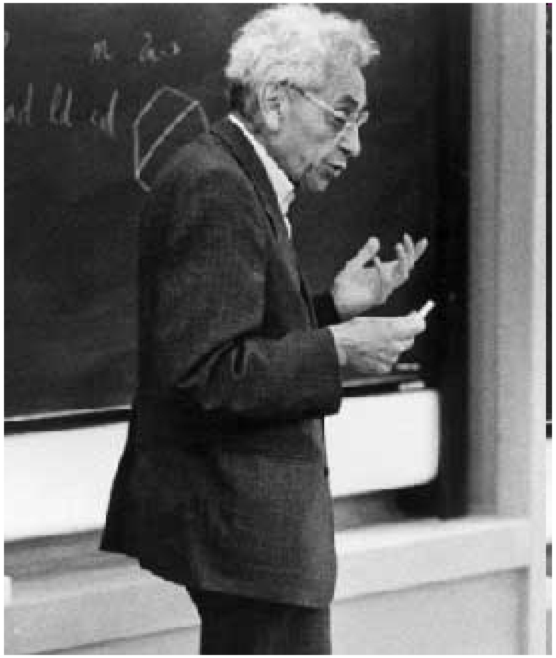
Four Colours Suffice - History

- 1852 Francis Guthrie -> Frederick Guthrie->Augustus de Morgan
- “Proof” by Alfred Kempe,1879
  - Disproved by Percy Heawood 1890
  - who then proved the “Five Colour Theorem”
- “Proof” by Peter Guthrie Tait 1880
  - Disproved by Julius Petersen 1891
- 1960s and 1970s Heinrich Heesch developed methods of using computers to search for a proof.
- Proved using a supercomputer 1976, by

Kenneth Appel and Wolfgang Haken (University of Illinois), with help on algorithmic work by John A. Koch

  - 1,476 configurations

Paul Erdös

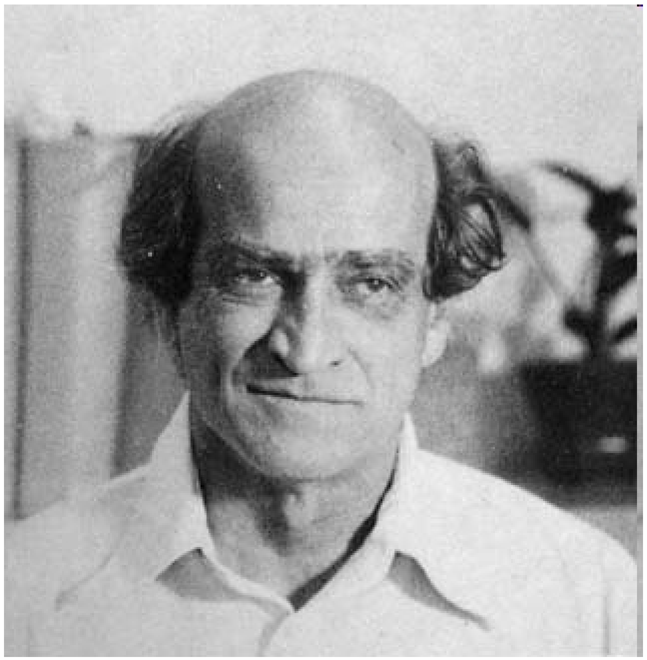


Born: 26 March 1913 in Budapest, Hungary
Died: 20 Sept 1996 in Warsaw, Poland

Itinerant mathematician

He alone had erdös number 0.

Ernst Gabor Straus



Born: 25 Feb 1922 in Munich, Germany
Died: 12 July 1983 in Los Angeles, California,

Erdös-Straus Conjecture

1948

The conjecture states that, for every integer $n \geq 4$, there exist positive integers *x*, *y*, and *z* such that

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

These unit fractions form an *Egyptian fraction* representation of the number $4/n$.

An *Egyptian fraction* is a representation of an irreducible fraction as a sum of unit fractions, as e.g. $5/6 = 1/2 + 1/3$.

Erdös-Straus Conjecture

```
>php erdoes-straus.php

4/5 = 1/2+1/4+1/20
4/5 = 1/2+1/5+1/10

4/7 = 1/2+1/18+1/63
4/7 = 1/2+1/21+1/42
4/7 = 1/2+1/28+1/28
4/7 = 1/3+1/6+1/14
4/7 = 1/4+1/4+1/14

4/9 = 1/3+1/10+1/90
4/9 = 1/3+1/12+1/36
4/9 = 1/3+1/18+1/18
4/9 = 1/4+1/6+1/36
4/9 = 1/4+1/9+1/12
4/9 = 1/6+1/6+1/9

4/11 = 1/3+1/66+1/66
4/11 = 1/4+1/11+1/44
4/11 = 1/4+1/12+1/33
4/11 = 1/6+1/6+1/33

4/13 = 1/4+1/26+1/52

4/15 = 1/5+1/18+1/90
4/15 = 1/5+1/20+1/60
4/15 = 1/5+1/24+1/40
4/15 = 1/5+1/30+1/30
4/15 = 1/6+1/12+1/60
4/15 = 1/6+1/14+1/35
4/15 = 1/6+1/15+1/30
4/15 = 1/6+1/20+1/20
```

Erdös-Straus Conjecture

Mordell gives some formulae. For instance

$$\frac{4}{n} = \frac{1}{n} + \frac{1}{(n-2)/3+1} + \frac{1}{n((n-2)/3+1)}.$$

whenever $n \equiv 2 \pmod 3$

>php mordell_1.php

4/5 = 1/5 + 1/2 + 1/10

4/8 = 1/8 + 1/3 + 1/24

4/11 = 1/11 + 1/4 + 1/44

4/14 = 1/14 + 1/5 + 1/70

4/17 = 1/17 + 1/6 + 1/102

4/20 = 1/20 + 1/7 + 1/140

4/23 = 1/23 + 1/8 + 1/184

Easily stated Conjectures

are not necessarily easy to prove.