TRANSCENDENTAL NUMBERS
THE GELFOND-SCHNEIDER THEOREM

Contents

TRANSCENDENTAL NUMBERS
THE GELFOND-SCHNEIDEE THEOREM

Refresher

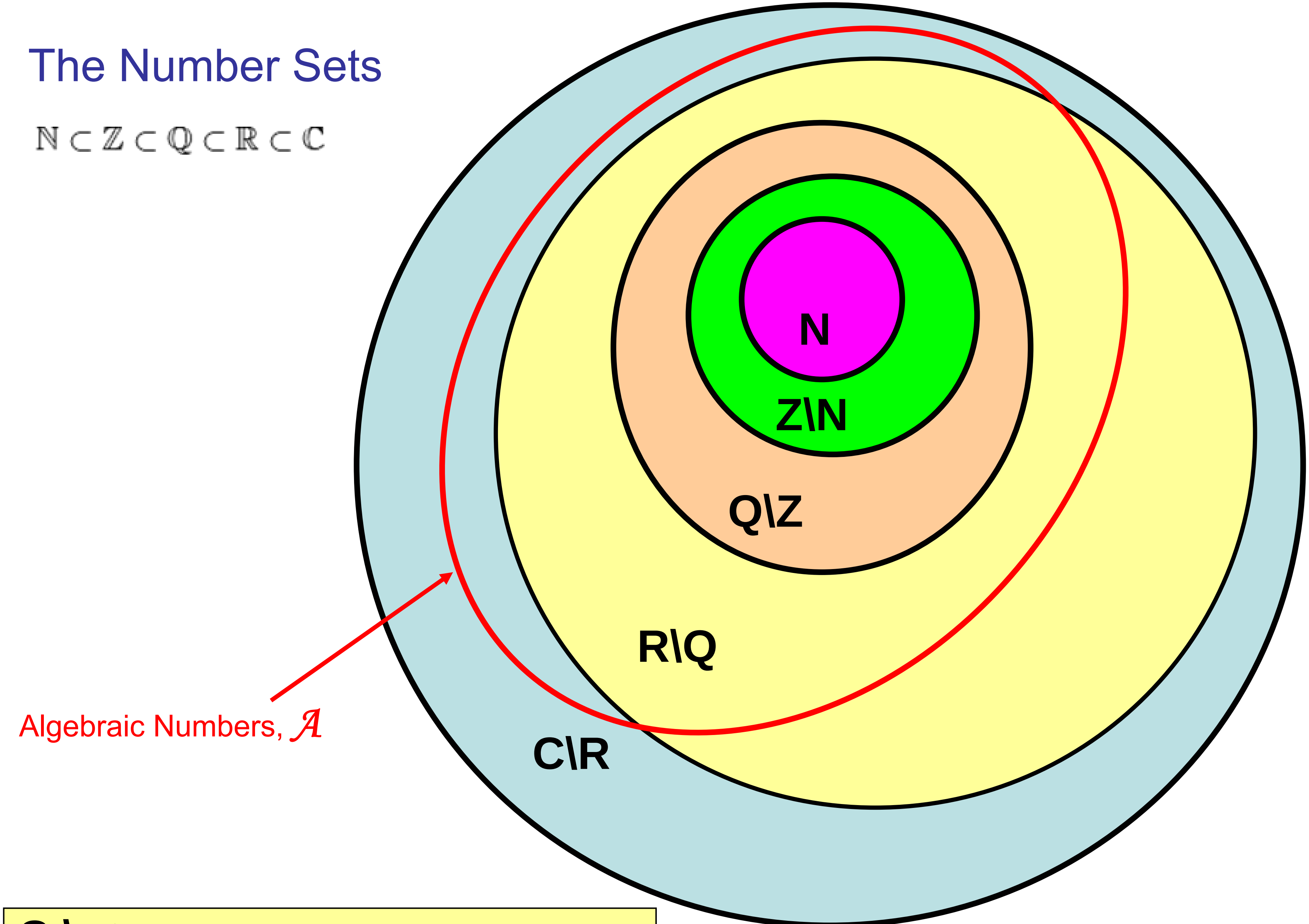
An **algebraic number** is a root of a polynomial in one variable with rational coefficients.
They were so called because the normal operations of **algebra** are used to find the roots (up to degree 4): +, -, x, ÷, √
Algebraic numbers can be +ve, -ve, rational, irrational, real, complex.

A **transcendental** number is one which is **not** algebraic.
Well known examples are: π. e, Ω (W(1)), Γ(1/4)

Φ (or ϕ) the *Golden Ratio* is irrational, but **not** transcendental because it is the solution of the polynomial: $x^2 - x - 1 = 0$

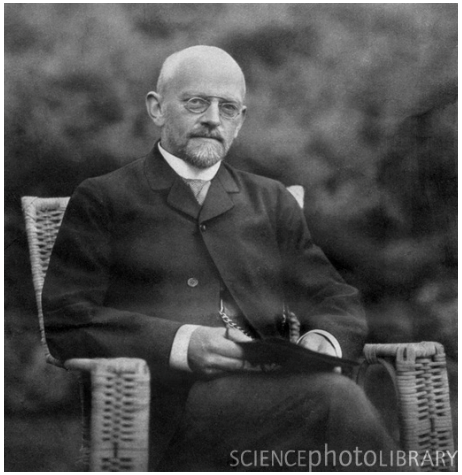
The Number Sets

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$



C \ $\mathcal{A}$ is the set of transcendental numbers.

HILBERT’S 23 PROBLEMS



1862, Königsberg – 1943, Göttingen

- David Hilbert presented a list of 23 problems at *The 2nd International Congress of Mathematicians* in 1900.
- Some were actually areas for investigation and not strictly problems.
- They were all unsolved or unexplored at the time.
- They were designed to serve as examples of the kinds of problems whose solutions would lead to the advancement of mathematics.

Resolved: 1, 3, 5, 7, 10, 14, 17, 18*, 19, 20, 21, 22. (* Computer-assisted proof.)

Partially resolved or no concensus: 2, 6, 9, 11, 12, 13, 15.

Too vague or broad to be declared as resolved or not: 4, 16, 23.

Not resolved at all: 8.

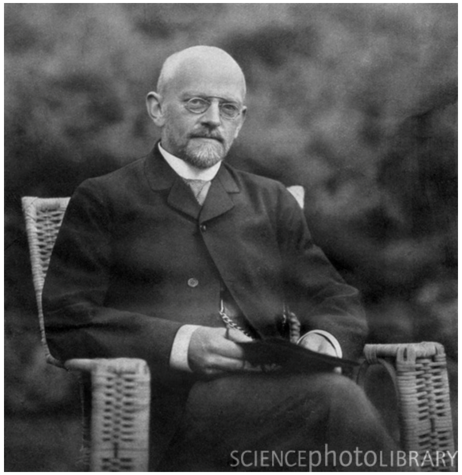
http://mathworld.wolfram.com/HilbertsProblems.html

Hilbert’s 23 Problems in Summary Form

1. Cantor's Problem of the Cardinal Number of the Continuum.
2. The Compatibility of the Arithmetical Axioms.
3. The Equality of the Volumes of Two Tetrahedra of Equal Bases and Equal Altitudes.
4. Problem of the Straight Line as the Shortest Distance Between Two Points.
5. Lie's Concept of a Continuous Group of Transformations Without the Assumption of the Differentiability of the Function Defining the Group.
6. Mathematical Treatment of the Axioms of Physics.
7. Irrationality and Transcendence of Certain Numbers.
8. Problems of Prime Numbers. (Riemann Hypothesis.)
9. Proof of the Most General Law of Reciprocity in any Number Field.
10. Determination of the Solvability of a Diophantine Equation.
11. Quadratic Forms With Any Algebraic Numerical Coefficients.
12. Extension of *Kronecker's Theorem* on Abelian Fields to Any Algebraic Realm. of Rationality
13. Impossibility of Solution of the General Equation of the 7th Degree by Means of Functions of Only Two Arguments.
14. Proof of the Finiteness of Certain Complete Systems of Functions.
15. Rigorous Foundations of Schubert's Enumerative Calculus.
16. Problem of the Topology of Algebraic Curves and Surfaces.
17. Expressions of Definite Forms by Squares.
18. Building up of Space from Congruent Polyhedra.
19. Are The Solutions of Regular Problems in the Calculus of Variations Always Necessarily Analytic?
20. The General Problem of Boundary Values.
21. Proof of the Existence of Linear Differential Equations Having a Prescribed Monodromic Group.
22. Uniformization of Analytic Relations by Means of Automorphic Functions.
23. Further Development of the Methods of the Calculus of Variations.

HILBERT'S 7TH PROBLEM

Hilbert's problems are a list of twenty-three problems presented by David Hilbert in 1900. They were all unsolved at the time.



1862, Königsberg – 1943, Göttingen

The seventh problem.

If α is algebraic and $\neq$ **0, 1** and β is algebraic and irrational is α^{β} transcendental?

This was proved independently by *Alexander Gelfond* and *Theodor Schneider* in 1934.

Gelfond and Schneider

Alexander Gelfond
1906, St Petersburg – 1968, Moscow

Theodor Schneider
1911, Frankfurt – 1988, Freiburg?

- **Alexander Gelfond** studied and taught at the Moscow State University.
- He studied briefly at Göttingen with Landau, Siegel and Hilbert.
- He was chief cryptographer to the Soviet Navy during WWII.

- **Theodor Schneider** solved Hilbert's 7^{th} problem in his PhD studies at Frankfurt.
- He worked at Göttingen, where he was an assistant to Carl Ludwig Siegel, and at Erlangen and Freiburg.

The Gelfond-Schneider Theorem

"If α and β are algebraic numbers and α is not **0** or **1** and β is irrational then α^{β} is transcendental."

Set notation form:

A is the set of algebraic numbers.

$$\alpha, \beta \in A \ , \ \alpha \neq 1, 0 \ , \ \beta \notin \mathbb{Q} \ \Rightarrow \ \alpha^{\beta} \notin A$$

i.e $\alpha^{\beta} \in \mathbb{C} \setminus A$

The set of algebraic numbers is infinite, so α^{β} can be varied infinitely. Thus the set of transcendental numbers is not small but infinite.

There are many more trancendental numbers than algebraic numbers.

Generalisation of the GST by Alan Baker

Alan Baker, FRS, 1939, London.
Trinity College, Cambridge
Fields Medal 1970

In the 1960's Alan Baker published several important papers on transcendence, including a generalization of the GST. One version of this is as follows.

If $\alpha_1, \alpha_2, \alpha_3 \ldots\ldots\ldots \alpha_n \in A, \neq 0,1$

and $\beta_1, \beta_2, \beta_3 \ldots\ldots\ldots \beta_n \in A, \notin \mathbb{Q}$

and $1, \beta_1, \beta_2, \beta_3 \ldots\ldots\ldots \beta_n$ **are linearly independent over the rationals.***

then $\alpha_1^{\beta_1}.\alpha_2^{\beta_2}.\alpha_3^{\beta_3} \ldots\ldots\ldots \alpha_n^{\beta_n} \notin A$ **i.e. is transcendental.**

Linearly Independent Numbers

A set of numbers is ***linearly independent over the rationals*** if it is not possible to multiply them by various rational coefficients such that they then sum to zero.

$$l_0.1 + l_1.\beta_1 + l_2.\beta_2 + l_3.\beta_3 + \ldots\ldots.l_n\beta_n \neq 0 \qquad l_i \in Q$$

The consequence of including “1” in the sequence above is that:

$$l_1.\beta_1 + l_2.\beta_2 + l_3.\beta_3 + \ldots\ldots.l_n\beta_n \notin \mathbb{Q}$$

Note 1) If β_1, β_2, β_3 …. were rational they would also be linearly ***dependent*** since it is always possible to find coefficients that make the above sum equal to zero.

Note 2) Two irrational numbers can sum to give a rational number.

Example (1). When there *is* linear dependence.

$$\alpha_1 = 5 \qquad \alpha_2 = 25 \qquad \beta_1 = \sqrt{3} \qquad \beta_2 = -\frac{\sqrt{3}}{2}$$

$$\alpha_1^{\beta_1} = 5^{\sqrt{3}} \qquad \alpha_2^{\beta_2} = 25^{\left(-\frac{\sqrt{3}}{2}\right)}$$

$$\beta_1 + 2 \times \beta_2 = \sqrt{3} + 2 \times \left(-\frac{\sqrt{3}}{2}\right) = 0$$

Thus β_1, β_2 are linearly *dependent*.

$$\alpha_1^{\beta_1} . \alpha_2^{\beta_2} = 5^{\sqrt{3}} . 25^{\left(-\frac{\sqrt{3}}{2}\right)}$$

$$5^{\sqrt{3}} . 25^{\left(-\frac{\sqrt{3}}{2}\right)} = 5^{\sqrt{3}} . 5^{\left(-\frac{\sqrt{3}}{2}\right)} . 5^{\left(-\frac{\sqrt{3}}{2}\right)}$$

$$5^{\sqrt{3}} . 5^{\left(-\frac{\sqrt{3}}{2}\right)} . 5^{\left(-\frac{\sqrt{3}}{2}\right)} = 5^{(0)} = 1$$

$$\alpha_1^{\beta_1} . \alpha_2^{\beta_2} \in \mathbb{Q} \in A$$

i.e. not transcendental.

Example (2). When there *is* linear dependence.

The sum of a rational and an irrational is irrational.

$$\left(1+\sqrt{2}\right), \left(1-\sqrt{2}\right), \frac{\left(1-\sqrt{2}\right)}{2} \notin \text{¤}$$

$$\alpha_1 = 5 \qquad \alpha_2 = 25 \qquad \beta_1 = \left(1+\sqrt{2}\right) \qquad \beta_2 = \frac{\left(1-\sqrt{2}\right)}{2}$$

$$\alpha_1{}^{\beta_1} = 5^{\left(1+\sqrt{2}\right)} \qquad \alpha_2{}^{\beta_2} = 25^{\frac{\left(1-\sqrt{2}\right)}{2}}$$

$$\left(-2\right)\times 1 + \beta_1 + 2\times\beta_2 = 0$$

Thus 1, β_1, β_2 are linearly dependent.

$$\alpha_1{}^{\beta_1}.\alpha_2{}^{\beta_2} = 5^{\left(1+\sqrt{2}\right)}.25^{\frac{\left(1-\sqrt{2}\right)}{2}}$$

$$5^{\left(1+\sqrt{2}\right)}.25^{\frac{\left(1-\sqrt{2}\right)}{2}} = 5^{\left(1+\sqrt{2}\right)}.5^{\frac{\left(1-\sqrt{2}\right)}{2}}.5^{\frac{\left(1-\sqrt{2}\right)}{2}} = 5^2 \in \text{¤} \in A$$

Can an Irrational Raised to an Irrational be Rational?
1) The Non-constructive Proof.

$$\alpha = \sqrt{2}^{\sqrt{2}} \qquad \beta = \sqrt{2}$$

$$\alpha^{\beta} = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$$

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}.\sqrt{2}} = \sqrt{2}^{2} = 2 \in \mathbb{Q}$$

β is known to be irrational.

Is α irrational?

Can an Irrational Raised to an Irrational be Rational? 1) A Non-constructive Proof.

$$\sqrt{2} \notin \mathbb{Q}$$

$\sqrt{2}^{\sqrt{2}}$ is either rational or irrational.

Option 1: $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$

In this case an irrational, √2, raised to an irrational,√2, is rational and we are done.

Option 2: $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$

We have shown that: $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = 2$

Therefore an irrational raised to an irrational is rational (2) and we are done.

We have therefore demonstrated that an irrational raised to an irrational can be rational.

This is a non-constructive proof because we did not actually show whether $\sqrt{2}^{\sqrt{2}}$ is rational or irrational.

Can an Irrational Raised to an Irrational be Rational?
1) A Constructive Proof.

$\alpha = \sqrt{2} \quad \sqrt{2} \notin \mathbb{Q}$

$\beta = \log_2 9 \qquad \log_2 9$ is either rational or irrational.

$$\alpha^{\beta} = \sqrt{2}^{(\log_2 9)} = \sqrt{2}^{\left(\log_2 3^2\right)} = \sqrt{2}^{2(\log_2 3)} = 2^{\log_2 3} = 3 \in \mathbb{Q}$$

We must now show that: $\beta \notin Q$ Let: $\beta = \dfrac{m}{n} \quad m, n \in \mathbb{Z}, n \neq 0$ i.e. $\beta \in Q$

$$\frac{m}{n} = \log_2 9$$

$m = n.\log_2 9 = \log_2 9^n \qquad m = \log_2 9^n \longrightarrow 2^m = 2^{\log_2 9^n} \longrightarrow 2^m = 9^n$

$2^m = 9^n$ This is impossible because 2^m is always even and 9^n is always odd.

Thus: $\log_2 9 \notin \mathbb{Q}$

Therefore we have demonstrated that an irrational raised to an irrational can be rational.

Is there a conflict with the Gelfond-Schneider Theorem?

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = 2 \qquad \sqrt{2}^{(\log_2 9)} = 3$$

In both cases : $\alpha^{\beta} \in \mathbb{Q}$

However, $\sqrt{2}^{\sqrt{2}}, \log_2 9 \notin A$

They are irrational, but also ***transcendental*.**

Thus if α or β is transcendental α^{β} ***may*** be rational.

What values of α and β lead to this result remains an open question.

Other case where α and/or β are transcendental.

$$e^{i\pi} = -1 \text{ *} \qquad e^{i\pi} = \left(e^{i}\right)^{\pi} \qquad \alpha^{\beta} \in \mathbb{Q} \subset A$$

α is the (complex) transcendental number e^i.

β is a transcendental number π.

If at least one of α or β is transcendental then α^β may be algebraic or transcendental.

z		w		z^w	
2	algebraic	$\log 3/\log 2$	transcendental	3	algebraic
2	algebraic	$i\log 3/\log 2$	transcendental	3^i	transcendental
e^i	transcendental	π	transcendental	-1	algebraic
e	transcendental	π	transcendental	e^π	transcendental
$2^{\sqrt{2}}$	transcendental	$\sqrt{2}$	algebraic	4	algebraic
$2^{\sqrt{2}}$	transcendental	$i\sqrt{2}$	algebraic	4^i	transcendental

Possibilities for z^w when z or w is transcendental.

(Marques and Sondow, 2010)

* See appendix

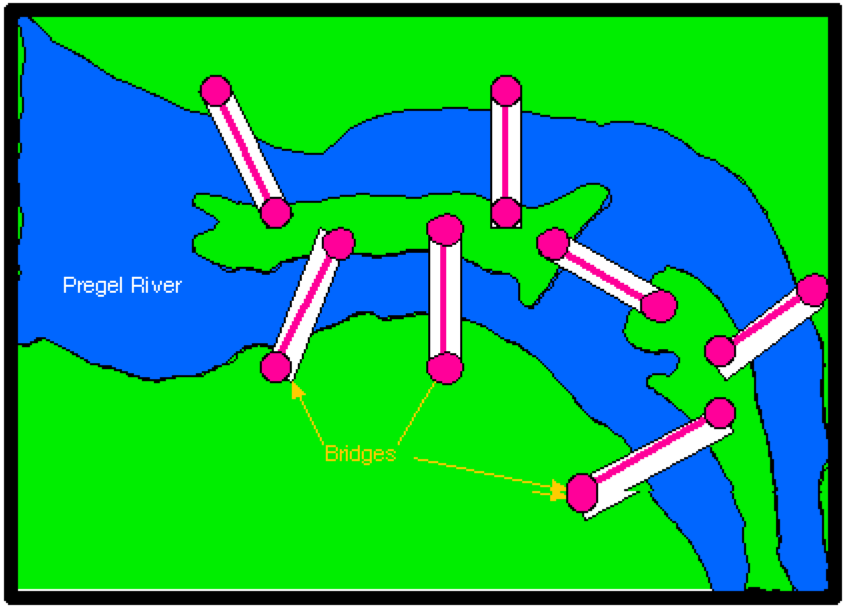
e^{π} Gelfond Constant

$2^{\sqrt{2}}$ Gelfond-Schneider Constant

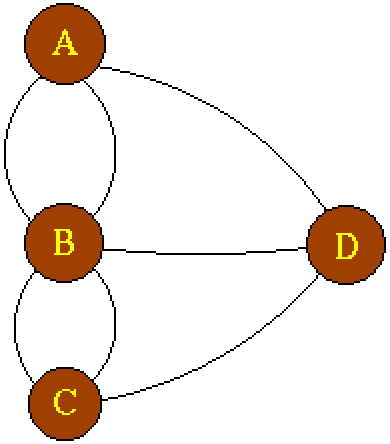
HILBERT’S SECOND GREAT LECTURE

KÖNIGSBERG, 1930

Königsberg, East Prussia / Kaliningrad, Russia

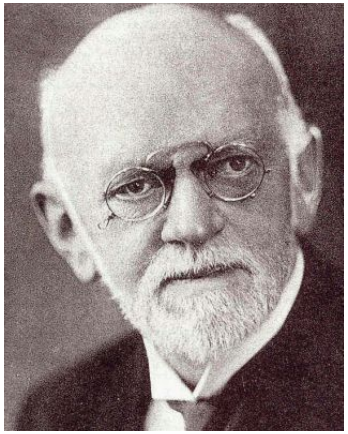


Leonhard Euler



Königsberg / Kaliningrad

Immanuel Kant

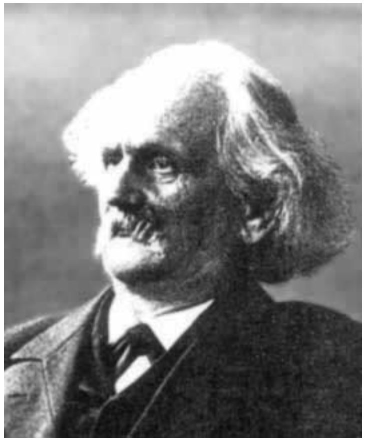


David Hilbert

?

Christian Goldbach

Carl Gustav Jacobi



Carl Gottfried Neumann

Hilbert’s Second Great Speech

On 8 September 1930 in Königsberg, at the *Congress of the Association of German Natural Scientists and Physicians*, David Hilbert gave a speech entitled *Naturerkennen und Logik* (Discovering Nature and Logic). A four minute excerpt was broadcast by radio, and has been preserved. (See the next slide.)

Hilbert used his speech to attack the theory promulgated by the psychologist Emil du Bois-Reymond and others that many things are unknowable and should not be studied. e.g. the nature of matter.

du Bois-Remond promoted the maxim:
ignoramus et ignorabimus, meaning “We do not know and shall not know."

Hilbert’s response was:
Wir müssen wissen. Wir werden wissen! (We must know. We shall know!)

Transcript of the radio broadcast.

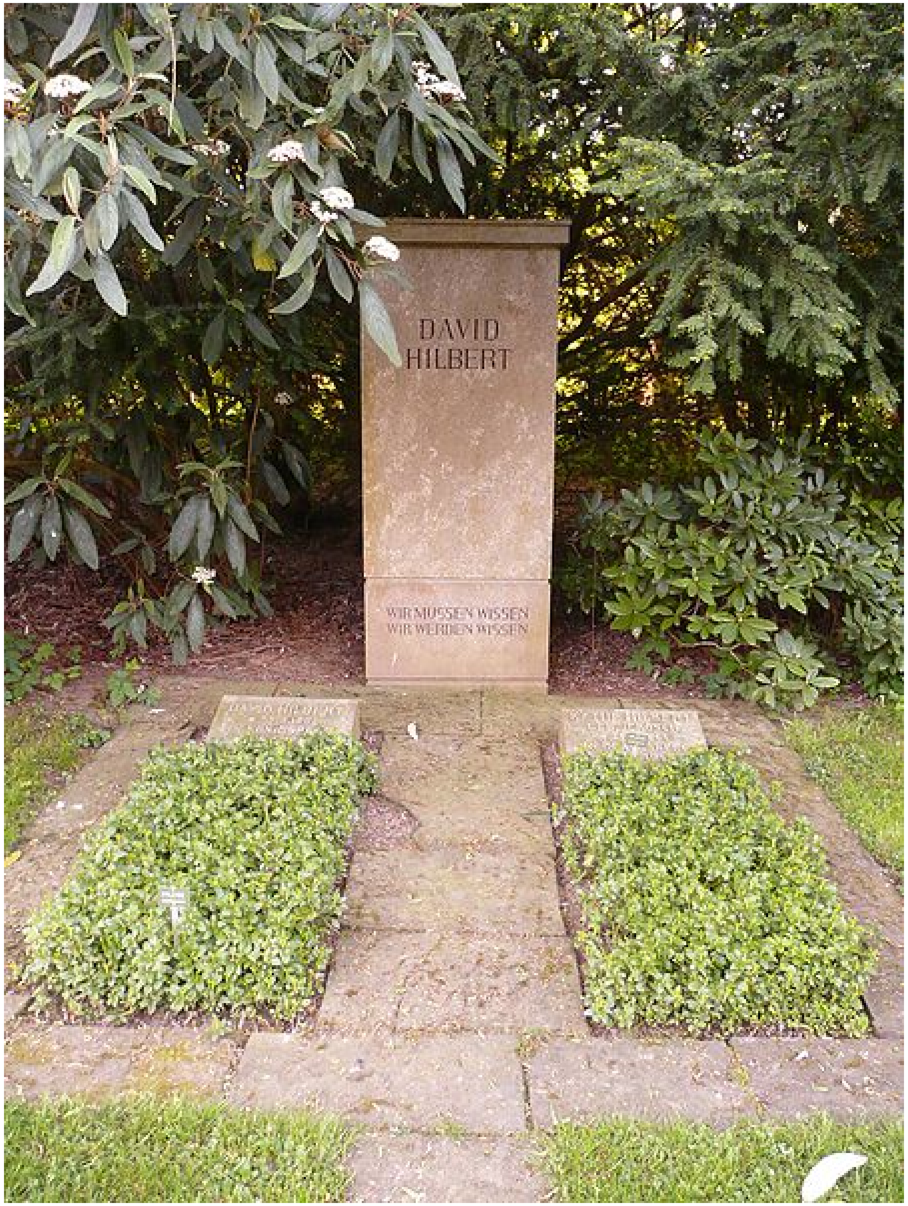
Das Instrument, welches die Vermittlung bewirkt zwischen Theorie und Praxis, zwischen Denken und Beobachten, ist die Mathematik; sie baut die verbindende Brückeund und gestaltet sie immer tragfähiger. Daher kommt es, daß unsere ganze gegenwärtige Kultur, soweit sie auf der geistigen Durchdringung und Dienstbarmachung der Naturberuht, ihre Grundlage in der Mathematik findet. Schon GALILEI sagt: Die Natur kannnur der verstehen der ihre Sprache und die Zeichen kennengelernt hat, in der sie zu uns redet; diese Sprache aber ist die Mathematik, und ihre Zeichen sind die mathematischen Figuren. KANT tat den Ausspruch: „Ich behaupte, daß in jeder besonderen Naturwissenschaft nur so viel eigentliche Wissenschaft angetroffen werden kann, als darin Mathematik enthalten ist.? In der Tat: Wir beherrschen nicht eher eine naturwissenschaftliche Theorie, als bis wir ihren mathematischen Kern herausgeschält und völlig enthüllt haben.Ohne Mathematik ist die heutige Astronomie und Physik unmöglich; diese Wissenschaften lösen sich in ihren theoretischen Teilen geradezu in Mathematik auf. Diese wie die zahlreichen weiteren Anwendungen sind es, den die Mathematik ihr Ansehen verdankt, soweit sie solches im weiteren Publikum genießt.
Trotzdem haben es alle Mathematiker abgelehnt, die Anwendungen als Wertmesser für die Mathematik gelten zu lassen. GAUSS spricht von dem zauberischen Reiz, den die Zahlentheorie zur Lieblingswissenschaft der ersten Mathematiker gemacht habe, ihres unerschöpflichen Reichtums nicht zu gedenken, woran sie alle anderen Teile der Mathematik so weit übertrifft. KRONECKER vergleicht die Zahlentheoretiker mit den Lotophagen, die, wenn sie einmal von dieser Kost etwas zu sich genommen haben, nie mehr davon lassen können. Der grosse Mathematiker POINCARÉ wendet sich einmal in auffallender Schärfe gegen TOLSTOI, der erklärt hatte, daß die Forderung „die Wissenschaft der Wissenschaft wegen? töricht sei. Die Errungenschaften der Industrie, zum Beispiel, hätten nie das Licht der Welt erblickt, wenn die Praktiker allein existiert hätten und wenn diese Errungenschaften nicht von uninteressierten Toren gefördert worden wären. Die Ehre des menschlichen Geistes, so sagte der berühmte Königsberger Mathematiker JACOBI, ist der einzige Zweck aller Wissenschaft.
Wir dürfen nicht denen glauben, die heute mit philosophischer Miene und überlegenem Tone den Kulturuntergang prophezeien und sich in dem Ignorabimus gefallen. Für uns gibt es kein Ignorabimus, und meiner Meinung nach auch für die Naturwissenschaft überhaupt nicht. Statt des törichten Ignorabimus heiße im Gegenteil unsere Losung:

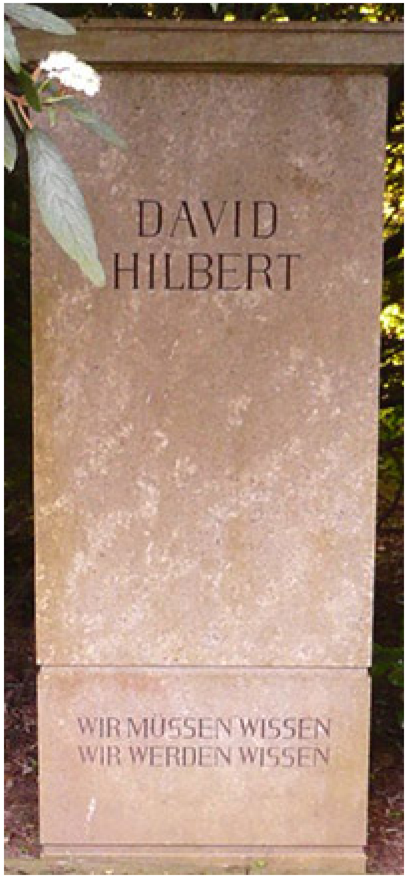
Wir müssen wissen.
Wir werden wissen!

The tool that serves as intermediary between theory and practice, between thought and observation, is mathematics. She builds the linking bridges, and gives them ever more reliable forms. Thus, it has come about that our contemporary culture, inasmuch as it is based on intellectual elucidation and the practical exploitation of nature, has its foundations in mathematics.
Already Galileo declared: "To understand nature, we must learn the language and the signs through which nature speaks to us." But this language is mathematics, and these signs are mathematical figures!
Kant pronounced: "I assert that, in every particular natural science, one encounters intrinsically scientific substance only to the extent that mathematics is present." Indeed, we do not master a scientific theory until we have shelled and completely pried free its mathematical kernel.
Without mathematics, the astronomy and physics of today would be impossible. These sciences, in their theoretical branches, virtually dissolve into mathematics. They, along with the many other applications, are responsible for whatever esteem mathematics enjoys with the general public.
Nevertheless, all mathematicians have refused to accept applications as valid measure of the worth of mathematics. Gauss speaks of the magical attraction that made number theory the darling discipline of the earliest mathematicians. Not to mention number theory's inexhaustible wealth, so far surpassing that of any other branch of mathematics.
Kronecker likens number theorists to the lotus eaters, who, having once savored this delight, could never give it up.
To Tolstoy, who had declared the pursuit of 'Science for the sake of Science' to be 'foolish', the great mathematician Poincaré replied with unusual sharpness, noting that the triumphs of industry, for example, would never have seen the light of day, if only practical men had existed, and these triumphs had not been made possible by disinterested 'fools'.
The glory of human intelligence, so said the famous Königsberg mathematician Jacobi, is the one purpose of all science.
We should not believe those who today prophesy the decline of scientific culture. Adopting a philosophical tone and an air of superiority, they smugly accept the concept of the 'unknowable'. For us mathematicians, there is no 'unknowable', and, in my opinion, there is none whatsoever for the natural sciences. In place of this foolish 'unknowable' let our watchword on the contrary be:

We must know.
We shall know!

Hilbert's Grave in Göttingen





Appendix

$$z = (\cos\theta + i\sin\theta)$$

$$\downarrow$$

$$\frac{dz}{d\theta} = (-\sin\theta + i\cos\theta)$$

$$\downarrow$$

$$(-\sin\theta + i\cos\theta) = i(i\sin\theta + \cos\theta) = iz$$

$$\downarrow$$

$$\frac{dz}{d\theta} = iz$$

$$\downarrow$$

$$\frac{dz}{z} = i\,d\theta$$

$$\downarrow$$

$$\int \frac{dz}{z} = \int i\,d\theta$$

$$\downarrow$$

$$\ln(z) = i\theta + c$$

$$z = e^{i\theta + c} = e^{i\theta}e^{c}$$

$$\downarrow$$

$$(\cos\theta + i\sin\theta) = e^{i\theta}e^{c}$$

$$\downarrow$$

$$\theta = 0, (1+0) = 1.e^{c} \Rightarrow c = 0$$

$$\downarrow$$

$$\boxed{(\cos\theta + i\sin\theta) = e^{i\theta}}$$

$$\downarrow$$

$$\theta = \pi, (\cos\pi + i\sin\pi) = e^{i\pi}$$

$$\downarrow$$

$$(-1+0) = e^{i\theta}$$

$$\downarrow$$

$$\boxed{e^{i\theta} = -1}$$